x_I = mole fraction of component I

Greek Letters

Θ_D = characteristic diffusion time
Θ_F = characteristic time of flow process
λ_m = characteristic or natural time of fluid
μ_0 = viscosity at zero shear rate
ρ_{1E} = final or equilibrium mass density of solvent in sample
ρ_{10} = initial mass density of solvent in sample

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Dissipation Effects in Hydrodynamic Stability of Viscoelastic Fluids

In this paper an analysis is made of the hydrodynamic stability of a Boussinesq viscoelastic fluid undergoing plane Couette flow with a superposed temperature gradient. Of special interest is the effect of including the dissipation term in the energy equation. This term is shown to destabilize the fluid for most values of disturbance wave number and material parameters and to cause overstability for all values of the Brinkman number. At a critical Weissenberg number of 1, a rheological instability is developed which is essentially independent of the Reynolds, Prandtl, and Brinkman numbers.

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SCOPE

One of the most common and important processing operations in the polymer industry is fiber spinning. This operation is depicted schematically in Figure 1 and consists of many sequential flows, each of which is important in the production of an end product of satisfactory material properties. For the purposes of the research described, the process will be divided heuristically into four regimes:

1. The die entry. Here the polymer melt is forced (by an extruder or other pressure source) from the reservoir into a capillary of very small diameter (relative to reservoir dimensions). The entry flow is very complicated and combines aspects of a complicated shear flow with an accelerated elongational flow.

2. The capillary. Here an approximation to simple shear flow is developed. Viscous dissipation induced temperature effects may be important.

3. Die exit. The complex phenomenon of die swell is encountered in this region.

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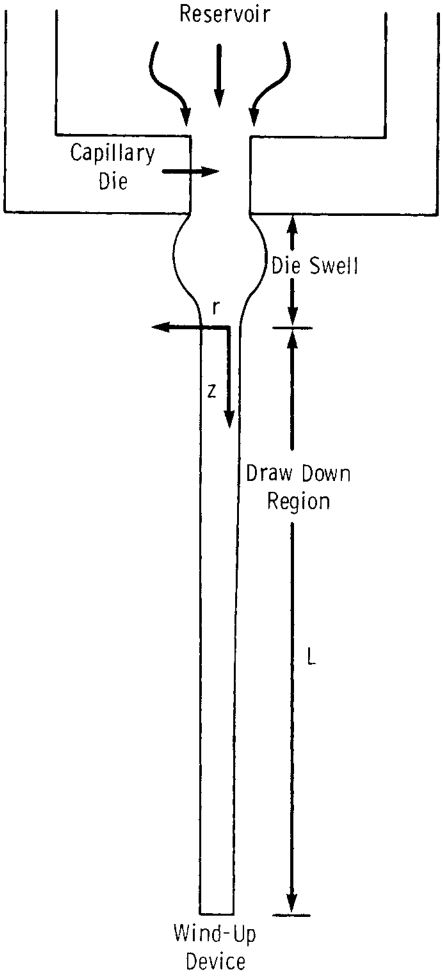

Fig. 1. Schematic diagram of the fiber spinning process.

4. Draw down. In this region the flow is primarily elongational as the fiber necks down from a maximum radius to the final desired fiber diameter. Cooling effects and crystallization of the polymer melt are of importance here.

Because of the complexity of the total problem, investigators trying to model the flows have normally limited themselves to one of the regimes. The work by Pearson (1966), Zamodits and Pearson (1969), and Martin (1970) on extruder flow and that of Ballenger and White (1970) and Han (1971) in the reservoir and capillary entrance region are very interesting.

In the capillary die, interest has centered around two effects and their relative importance and/or interrelationship: viscous heating and melt fracture. With regard to viscous dissipation effects, important contributions have been made by Joseph (1965), Bird and Turian (1962, 1965), Martin (1967), Sukanek and Lawrence and coworkers (1968, 1971, 1973, 1974), Trowbridge and Karran (1973), Cox and Macosko (1974), and others. Both the capillary and die inlet regions have been postulated as initiation sites for the flow instability known as melt fracture (Tordella, 1963). This will be discussed in detail below.

Probably the flow region which has received the most attention recently has been the draw down regime. This is usually modeled as an extensional or elongational flow. Predictions of the stress extension rate behavior of polymer melts vary greatly, depending on the constitutive relation and exact form of the kinematics assumed (see Coleman and Noll 1962; Tanner, 1969; Astarita and Nicodemo, 1970; Dealy, 1971; Marrucci and Murch, 1970; Stevenson and Bird, 1971; Stevenson, 1972; and Cogswell, 1972).

The application of these predictions to the actual description of melt spinning, both experimentally and theoretically, has resulted in the interesting and important work of Matovich and Pearson (1969), Moore and Pearson (1972), Spearot and Metzner (1972), Chen et al. (1972), Han (1970), and Han and Lamonte (1972). Temperature effects due to cooling and crystallization of polymer from the melt (Dees and Spruiell, 1974) are important in this region but are normally not included in the mathematical modeling studies because of the complexities introduced.

The research described in this paper is concerned with the explanation of an important melt processing problem mentioned above, melt fracture. Melt fracture analysis is concerned with the origin of the development of waviness or matte on the surface of melt extrudate as it leaves the capillary die (see Tordella, 1963, 1957; Bagley et al. 1963; Ballenger et al., 1971). In extreme cases the extrudate can have the appearance of being shattered into pieces, thus the name melt fracture. Extensive studies of this phenomenon have been carried out recently by Vinogradov et al. (1972), Han and Lamonte (1971, 1972), Vlachopoulos (1972), Boger and Williams (1972), Ballenger et al. (1971), Sadd (1973), and Everage and Ballman (1974). Most of the effort has been to develop physically based but empirical correlations of melt physical and viscometric properties with onset of melt fracture. The most successful has been the critical Weissenberg number (ratio of first normal stress difference to shear stress) or critical recoverable shear strain criteria (these are basically equivalent). We have used classical hydrodynamic stability analysis to investigate quantitatively the origin of this industrially important problem. The results obtained are encouraging and give a physical basis for the empirically based correlations.

CONCLUSIONS AND SIGNIFICANCE

This work extends the recent research of McIntire (1972), and Rothenberger et al. (1973) to show that simple shearing flows of viscoelastic fluids develop a hydrodynamic instability, due purely to the rheological properties of the material, at a critical value of Weissenberg number. The actual critical value is close to that developed from empirical correlations. This gives confidence that classical hydrodynamic stability analyses of industrially important flows may lead to extremely useful understanding of the underlying causes of processing problems. It should be emphasized that constitutive models used should allow viscoelastic behavior, since these features are often the critical phenomena in the stability analyses (that is, power law fluids, etc., will not do).

The critical Weissenberg number is essentially independent of the Reynolds and Prandtl numbers. Allowance for viscous dissipation effects (nonzero Brinkman number) introduces overstability for all values of critical Rayleigh number. The value of the Weissenberg number at which the flow becomes unstable at zero Rayleigh number (no temperature gradient) is independent of Brinkman number.

This mechanism of instability is put forward as at least one, if not the only, cause of melt fracture initiation. The critical wavelength of the disturbances is very small, in agreement with experimental observations. The instability would be expected to be initiated near the wall in Poiseuille flow, as this is the region of highest shear rate, and thus highest Weissenberg number.

It should be emphasized that two modes of instability are possible in the model flow system considered. One is the convective mode, characterized by small critical wave number, and this is seen for Weissenberg numbers of less than 1. Since the critical Rayleigh number is of order 10^3 and typical Rayleigh numbers in polymer processing are of order 10, it is unlikely that convective instabilities will be important for polymer melts. The second mode, characterized by large wave number critical disturbances, is of rheological origin and can occur with a zero Rayleigh number if the Weissenberg number exceeds 1. This is the mode which is of principal importance in polymer melts.

This paper concerns the effect of viscous heating on the hydrodynamic stability of viscoelastic fluids. Most of the literature on hydrodynamic stability of fluids deals with systems in which the dissipation terms in the energy equation are neglected.

Most workers who have dealt with dissipation effects have not dealt with hydrodynamic stability. Brinkman (1952), Nahme (1940), Kearsley (1962), and Bird and Turian (1962) have considered Newtonian fluid flows with dissipation. Turian (1965, 1969), Martin (1967), and Gavis and Laurence (1968) have looked at dissipation effects on the flow of inelastic non-Newtonian fluids. The hydrodynamic stability of Newtonian fluids, including dissipation and temperature dependent viscosity effects, has been considered mainly by Joseph (1964, 1965), Joseph and Warner (1967), Sukanek et al. (1973), and Lebon and Nguyen (1974). There appears to have been no work done on the stability of viscoelastic fluids with dissipation.

In polymer processing, high-speed flows of nonisothermal viscoelastic fluids are common in melt operations. In extrusion processes, polymer melt flow instabilities are common and must be avoided. The literature in this area has recently been reviewed by Ballenger et al. (1971), and emphasis is put on the role of the Weissenberg number (ratio of first normal stress difference to shear stress) in the initiation of the instability called *melt fracture*.

To model this flow we have considered plane Couette flow with superposed temperature gradient (McIntire and Schowalter, 1970, 1972). This extends the generality of the results to include instabilities due to thermal convection as well as rheological parameters. The effect of the Weissenberg number on the stability criteria is very provocative. Also, the effect of the dissipation terms on the principle of exchange of stabilities or overstability is interesting.

BALANCE EQUATIONS

The mass, momentum, and thermal energy balances, including the change in energy due to viscous dissipation, are given by

Mass

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

Momentum

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\, \mathbf{v}) + \nabla p = \nabla \cdot \boldsymbol{\tau} + \rho \mathbf{g} \tag{2}$$

Energy

$$\rho C_v \frac{DT}{Dt} + \nabla \cdot \mathbf{q} + T \left(\frac{\partial p}{\partial T}\right)_V (\nabla \cdot \mathbf{v}) = \boldsymbol{\tau} : \nabla \mathbf{v} \tag{3}$$

These equations will be simplified under the following assumptions, which are used throughout the paper.

1. The Boussinesq approximation is made. This implies that all physical parameters are assumed constant with respect to temperature, except the density in the buoyancy term of the momentum balance. In addition, all physical parameters are assumed constant in space and time (no degradation effects).
2. A Fourier type of heat conduction vector ($\mathbf{q}$) is assumed, by neglecting the nonconlinearity of heat flux and temperature gradient that Huilgol (1969) demonstrates for the second-order fluid.
3. All dependent variables (p, ρ, T, $\mathbf{v}$, $\boldsymbol{\tau}$, $\mathbf{q}$) are perturbed, and the perturbations are assumed sufficiently small so that nonlinear combinations of perturbations and their derivatives are negligible (see Chandrasekhar, 1961, for a definitive treatment of linearized stability theory).

The following perturbations are introduced into the balance equations:

$$\begin{aligned} \mathbf{v}' &= \overline{\mathbf{V}} + \mathbf{u} & p' &= \overline{p} + \delta p \\ T' &= \overline{T} + \theta & \tau' &= \overline{\boldsymbol{\tau}} + \delta \boldsymbol{\tau} \\ \rho' &= \overline{\rho} + \delta\rho & \mathbf{q}' &= \overline{\mathbf{q}} + \delta \mathbf{q} \end{aligned} \tag{4}$$

By noting that the steady state balance equations are satisfied by the steady state variables (denoted by the raised bar), these steady state terms are deleted from the balance equations. The mass balance gives

$$\nabla \cdot \mathbf{v}' = \nabla \cdot \overline{\mathbf{V}} = \nabla \cdot \mathbf{u} = 0 \tag{5}$$

This result is used in simplifying the remaining two balances. When we observe that the material derivative of the steady state variables vanishes and that the Boussinesq approximation implies that $\rho' = \rho_0$ in every term but the $\delta \rho g$ term, the linearized perturbation is given by

$$\rho_0 \left\{ \frac{\partial \mathbf{u}}{\partial t} + (\overline{\mathbf{V}} \cdot \nabla)\, \mathbf{u} + (\mathbf{u} \cdot \nabla)\, \overline{\mathbf{V}} \right\} + \nabla (\delta p) = \nabla \cdot \delta \boldsymbol{\tau} + \delta \rho \mathbf{g} \tag{6}$$

$$\rho_0 C_v \left\{ \frac{\partial \theta}{\partial t} + (\overline{\mathbf{V}} \cdot \nabla)\, \theta + (\mathbf{u} \cdot \nabla)\, \overline{T} \right\} + \nabla \cdot \delta \mathbf{q} = \overline{\boldsymbol{\tau}} : \nabla \mathbf{u} + \delta \boldsymbol{\tau} : \nabla \overline{\mathbf{V}} \tag{7}$$

CONSTITUTIVE RELATION

Two approximations to the Coleman and Noll (1961) simple fluid are commonly used to model the stress response of viscoelastic materials. The first is the small motion approximation which leads to simple integral constitutive relations. A particular example of one of these models is used in this paper. The form was introduced by Bird and Carreau (1968) and has the advantage of allowing two normal stress differences and a shear dependent viscosity in simple shearing flows, properties exhibited by real polymer solutions and melts. The integral model is described by

$$\boldsymbol{\tau} = \mathbf{S} + p\mathbf{I} = \int_{-\infty}^{t} m\,[(t - t'),\, II(t')] \left\{ \left(1 + \frac{\delta}{2}\right) \mathbf{B} + \frac{\delta}{2} \mathbf{C} \right\} dt' \tag{8}$$

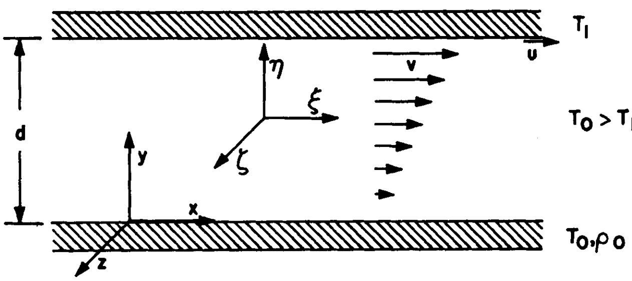

Fig. 2. The Physical Situation—Plane Couette Flow with superposed temperature gradient. Lower plate at constant temperature T_0. Upper plate moving with velocity U and at constant tempeature T_1.

where

$$m\{(t-t'),\, II(t')\} = \sum_{p=1}^{\infty} m_p \exp\left[\frac{-(t-t')}{\lambda_{2p}}\right]$$

$$m_p = \frac{\eta_p}{\lambda_{2p}^2\,[1 + 2\,II(t')\,\lambda_{1p}^2]}$$

$$\mathbf{C} = \mathbf{i}_j\left(\frac{\partial x_\alpha'}{\partial x_j}\frac{\partial x_\alpha'}{\partial x_k} - \delta_{jk}\right)\mathbf{i}_k \qquad \delta_{jk} = \begin{Bmatrix} 1 & j=k \\ 0 & j\neq k \end{Bmatrix}$$

$$\mathbf{B} = \mathbf{i}_j\left(\frac{\partial x_j}{\partial x_\alpha'}\frac{\partial x_k}{\partial x_\alpha'} - \delta_{jk}\right)\mathbf{i}_k$$

In this paper the m_p are assumed constant, which implies that variations in $II(t')$ due to velocity perturbations are negligible in plane Couette flow. The strain tensors **B** and **C** are the modified Finger and Green strain tensors, respectively, written in rectangular Cartesian coordinate notation. This model has been used successfully in similar hydrodynamic stability problems (McIntire, 1970, 1971).

The second commonly used constitutive approximations are the slow or retarded motion expansions (Coleman and Noll, 1960) which lead to the second-order and higher-order fluids. These models have been known to give ambiguous results in time dependent motions (Coleman et al., 1965), and hydrodynamic stability analyses employing these models must be done with great care (Craik, 1968; McIntire, 1971). Only the integral constitutive model is used in this paper.

PHYSICAL SITUATION

The physical problem to be considered is a fluid in plane Couette flow, heated from below. The coordinate system (see Figure 2) is such that the y coordinate spans the gap between two parallel plates, infinite in the x and y directions and at different constant temperatures. The positive x direction is the direction that the top plate moves at constant velocity U. The steady state velocity field is given by

$$\overline{\mathbf{V}} = \left(\frac{Uy}{d}, 0, 0\right)$$
$$U = \dot\gamma d \tag{9}$$

$\dot\gamma$ is the shear rate $d\overline{V}/dy$. The steady state stress tensor is easily calculated to be

$$\mathbf{T} = \begin{bmatrix} \beta_1\left(1+\frac{\delta}{2}\right)\dot\gamma & \beta_0 & 0 \\ \beta_0 & \beta_1\frac{\delta\dot\gamma}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}\dot\gamma \tag{10}$$

where

$$\beta_0 = \sum_{p=1}^{\infty} m_p\lambda_{2p}^2$$

$$\beta_1 = \sum_{p=1}^{\infty} 2m_p\lambda_{2p}^3$$

Because this paper deals with dissipation as a factor in the equations, the temperature profiles are not linear, but parabolic:

$$\overline{T} - T_0 = \frac{-\pi y^2}{2} + \left(\beta_T + \frac{\pi d}{2}\right)y \tag{11}$$

$$\pi = \frac{\beta_0 U^2}{kd^2}$$

k = thermal conductivity

$$\beta_T = \frac{T_1 - T_0}{d}$$

STRESS PERTURBATION

In calculating the Finger strain tensor for the perturbed motion, it is necessary to rely heavily on the linearization assumption. We have assumed that the steady state velocity $\overline{\mathbf{V}}$ is perturbed: $\mathbf{v}' = \overline{\mathbf{V}} + \mathbf{u}$. Let the perturbation velocity components be defined by $u_i = u_i(\mathbf{x})h(t)$, where all the time dependency is in $h(t)$. The normal mode decomposition (see below) gives $h(t) = e^{\sigma t}$, where σ is a complex constant. Using the linearization approximation and a large amount of algebra, we can show that (Bonnett, 1972)

$$\delta\mathbf{T} = \int_{-\infty}^{t}\{m\,[(t-t'),\, II(t')]\}\left[\left(1+\frac{\delta}{2}\right)(\delta\mathbf{B}) + \frac{\delta}{2}(\delta\mathbf{C})\right]dt'$$

$$\delta\tau_{xx} = \left(1+\frac{\delta}{2}\right)\left\{2\dot\gamma^3\mu_4\frac{\partial u_y}{\partial x} + 2\dot\gamma\mu_2\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) - \dot\gamma^2\mu_3\frac{\partial u_z}{\partial x}\right\} + 2\mu_1\frac{\partial u_x}{\partial x}$$

$$\delta\tau_{yy} = \left(1+\frac{\delta}{2}\right)2\dot\gamma\mu_2\frac{\partial u_y}{\partial x} + 2\frac{\partial u_y}{\partial y}\mu_1 + 2\delta\dot\gamma\frac{\partial u_x}{\partial y}$$

$$\delta\tau_{zz} = 2\mu_1\frac{\partial u_z}{\partial z}$$

$$\delta\tau_{xy} = \left(1+\frac{\delta}{2}\right)\left\{2\dot\gamma^2\mu_3\frac{\partial u_y}{\partial x} + \mu_2\dot\gamma\left(\frac{\partial u_y}{\partial y} - \frac{\partial u_z}{\partial z}\right)\right\} + \mu_1\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{\delta}{2}\mu_2\dot\gamma\frac{\partial u_x}{\partial x} \tag{12}$$

$$\delta\tau_{xz} = \left(1+\frac{\delta}{2}\right)\left\{\dot\gamma\mu_2\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \dot\gamma^2\mu_3\frac{\partial u_z}{\partial x}\right\} + \mu_1\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\delta\tau_{yz} = \left(1+\frac{\delta}{2}\right)\left\{\dot\gamma\mu_2\frac{\partial u_z}{\partial x}\right\} + \mu_1\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) + \frac{\delta\dot\gamma}{2}\mu_2\frac{\partial u_x}{\partial z}$$

where

$$\mu_1 = \sum_p \frac{\lambda_{2p}{}^2 m_p}{1+\sigma\lambda_{2p}}$$

$$\mu_2 = \sum_p \frac{\lambda_{2p}{}^3 (2+\sigma\lambda_{2p}) m_p}{(1+\sigma\lambda_{2p})^2} \tag{13}$$

$$\mu_3 = \sum_p \frac{2\lambda_{2p}{}^4 \left[3 + 3\sigma\lambda_{2p} + (\sigma\lambda_{2p})^2\right] m_p}{(1+\sigma\lambda_{2p})^3}$$

$$\mu_4 = \sum_p \frac{6\lambda_{2p}{}^5 \left[4 + 6\sigma\lambda_{2p} + 4(\sigma\lambda_{2p})^2 + (\sigma\lambda_{2p})^3\right] m_p}{(1+\sigma\lambda_{2p})^4}$$

Nondimensional Equations: Normal Mode Analysis

The pressure term in the momentum balance is eliminated by taking the curl of Equation (6). To return to the desired primary variables, the curl is taken once again. The stress perturbations are then substituted into the modified Equations (6) and (7) (Bonnett, 1972). At this point it is convenient to introduce dimensionless variables and to shift the origin from the stationary bottom plate to the middle of the gap, which is now assumed stationary. Physically, what is done can be explained by saying that the bottom plate now has velocity $-U/2$ and the top plate $+U/2$. This leaves the velocity gradient, and therefore $\dot{\gamma}$, unchanged. The temperature profile at steady state is algebraically changed, however. The following dimensionless variables are used:

$$\begin{aligned}
\frac{\mathbf{u}}{U} &= (u_\xi, u_\eta, u_\zeta) \\
\xi &= \frac{x}{d} \\
\eta &= \left(\frac{y}{d} - \frac{1}{2}\right) \\
\zeta &= \frac{z}{d} \\
\psi &= \frac{\theta}{T_o - T_1} = \frac{\theta}{\Delta T} \\
\frac{\partial}{\partial t} &= \frac{d}{U}\frac{\partial}{\partial t} \\
U &= \dot{\gamma} d
\end{aligned} \tag{14}$$

The steady state temperature profile becomes, defining the Brinkman number, $Br = \beta_o U^2/k\Delta T$:

$$\frac{\partial T}{\partial \eta} = \frac{d}{\Delta T}\left(\frac{\partial \overline{T}}{\partial y}\right) = -\left[1 + Br\eta\right] \tag{15}$$

The assumption is now made that the disturbance velocities can be written in terms of normal modes (see Chandrasekhar, 1961):

$$\begin{aligned}
u_\xi &= \frac{i}{v_x \pm v_z}\frac{dv(\eta)}{d\eta} \exp\left[i(v_x\xi + v_z\zeta) + \sigma t\right] \\
u_\eta &= v(\eta) \exp\left[i(v_x\xi + v_z\zeta) + \sigma t\right] \\
u_\zeta &= \frac{\pm i}{v_x \pm v_z}\frac{dv(\eta)}{d\eta} \exp\left[i(v_x\xi + v_z\zeta) + \sigma t\right] \\
\psi &= \psi(\eta) \exp\left[i(v_x\xi + v_z\zeta) + \sigma t\right]
\end{aligned} \tag{16}$$

In Equations (16) the wave numbers v_x and v_z are real constants, and σ is a complex growth parameter. This study uses (arbitrarily) the minus sign in u_ξ and u_ζ and does not look at the case $v_x = v_z$. The perturbation equations are not immediately amenable to solution with the modified Galerkin technique (Finlayson, 1968). A simplification is made by assuming $\sigma << 1$ near the neutral stability curve. Terms $0(\sigma^2)$ and smaller are neglected.

With the above manipulations and substitutions, and by introducing a new dimensionless temperature

$$\theta = \frac{\alpha g \Delta T d^2 \rho_o K^2}{\beta_o U}\psi(\eta)$$

the momentum and energy balances become the following:
Momentum

$$\begin{aligned}
&\sigma\Bigg\{ Re\nabla^2 v - \lambda\Bigg[\nabla^4 v - 2\,Rei v_x \eta\nabla^2 v \\
&+ i\left(v_x A - \frac{K^2 B}{v_x - v_z}\right)\nabla^2\frac{dv}{d\eta} - 24\,Div_x{}^3\frac{dv}{d\eta} - 2\theta\Bigg]\Bigg\} \\
&= \nabla^4 v - Rei\,v_x\eta\nabla^2 v + 2i\left(v_x A - \frac{K^2 B}{v_x - v_z}\right)\nabla^2\frac{dv}{d\eta} \\
&+ 12\,Cv_x{}^2\left(\frac{d^2v}{d\eta^2} + K^2 v\right) + 48\,Div_x{}^3\frac{dv}{d\eta} - \theta
\end{aligned} \tag{17}$$

Energy

$$\begin{aligned}
&\sigma\Bigg\{Pe\theta - \lambda\Bigg[2RaK^2(1 + Br\eta)\,v - 2Peiv_x\eta\theta \\
&+ 2\nabla^2\theta + \frac{RaBrK^2}{Pe}\Bigg[\left(\frac{A(v_x - 2v_z) - Bv_x}{v_x - v_z}\right)\frac{dv}{d\eta} \\
&+ 3i\left(\frac{1}{v_x - v_z}\frac{d^2v}{d\eta^2} + v_x v\right)\Bigg]\Bigg]\Bigg\} \\
&= RaK^2(1 + Br\eta)\,v - Peiv_x\eta\theta + \nabla^2\theta \\
&+ \frac{RaBrK^2}{Pe}\Bigg\{12iCv_x v \\
&\qquad + \frac{A(v_x - 4v_z) - 2B(v_x + v_z)}{v_x - v_z}\frac{dv}{d\eta} \\
&+ 2i\left[\frac{\frac{d^2v}{d\eta^2}}{v_x - v_z} + v_x v\right]\Bigg\}
\end{aligned} \tag{18}$$

where

$$Re = \text{Reynolds number} = \frac{\rho_o \dot{\gamma} d^2}{\beta_o} \qquad K^2 = v_x{}^2 + v_z{}^2$$

$$Pe = \text{Peclet number} = \frac{\rho_o C_v \dot{\gamma} d^2}{k} \qquad \lambda = \sum_{p=1}^{\infty}\lambda_{2p}$$

$$Ra = \text{Rayleigh number} = \frac{\alpha g\,\nabla T d^3 \rho_o{}^2 c_v}{k\beta_o}$$

$$A = \left[\sum_{P=1}^{\infty} m_p\lambda_{2p}{}^3 \Big/ \sum_{p=1}^{\infty} m_p\lambda_{2p}{}^2\right]\left(1 + \frac{\delta}{2}\right)\dot{\gamma}$$

$$B = \left[\sum_{P=1}^{\infty} m_p\lambda_{2p}{}^3 \Big/ \sum_{p=1}^{\infty} m_p\lambda_{2p}{}^2\right]\frac{\dot{\gamma}\delta}{2}$$

$$C = \left[\sum_{p=1}^{\infty} m_p \lambda_{2p}^4 \Big/ \sum_{p=1}^{\infty} m_p \lambda_{2p}^2 \right] \left(1 + \frac{\delta}{2}\right) \dot{\gamma}^2$$

$$D = \left[\sum_{p=1}^{\infty} m_p \lambda_{2p}^5 \Big/ \sum_{p=1}^{\infty} m_p \lambda_{2p}^2 \right] \left(1 + \frac{\delta}{2}\right) \dot{\gamma}^3$$

The boundary conditions for the velocity and temperature perturbations are

$$\begin{aligned} v(\eta) &= 0 \quad \text{at} \quad \eta = \pm\frac{1}{2} \\ \theta(\eta) &= 0 \quad \text{at} \quad \eta = \pm\frac{1}{2} \\ \frac{dv}{d\eta} &= 0 \quad \text{at} \quad \eta = \pm\frac{1}{2} \end{aligned} \tag{19}$$

The first two sets of conditions imply that the channel walls are at constant temperature and velocity. The third set comes from the no slip condition and the continuity equation.

COMPARISON WITH PREVIOUS WORK

It is instructive to simplify the above equations to show correspondence of these equations with those of other workers. The first simplification is to assume $Br = 0$, since other workers have not considered dissipation effects in this type of hydrodynamic stability analysis. If $Br = 0$, and if all the non-Newtonian parameters are zero [$A = B = C = D = \lambda = 0$], the equations of Deardorff (1965) and Galleghar and Mercer (1965) are recovered:

$$\begin{aligned} Re\,[\sigma \nabla^2 v + i v_x \eta\, \eta \nabla^2 v] &= \nabla^4 v - \theta \\ Pe\,[\sigma\theta + i v_x\, \eta\theta] &= RaK^2 v + \nabla^2 \theta \end{aligned} \tag{20}$$

If the fluid is viscoelastic, but $Br = 0$, $v_x = 0$, and the exchange of stabilities assumption (see Chandrasekhar, 1961) is made, the equations reduce to those of McIntire and Schowalter (1970). The latter authors also examined (1972):

1. The case $v_x \neq 0$, but exchange of stabilities valid.
2. The case $v_x = 0$, but exchange of stabilities not assumed.

These are all special cases of the above equations.

NUMERICAL SOLUTION

For the purpose of analysis, the left-hand sides of Equations (17) and (18) are regarded as being differentiated with respect to time, instead of multiplied by the complex growth rate σ. This is an assumption that will be valid if the modified Galerkin technique produces distinct eigenvalues. Numerical work indicates that this is the case.

The Galerkin method used in this paper is essentially that of Finlayson (1968) and has been used successfully by others in similar hydrodynamic stability problems (McIntire and Schowalter, 1972). The systems of equations have two unknown functions $v(\eta, t)$ and $\theta(\eta, t)$. The sets of approximating functions are chosen to satisfy the boundary conditions exactly. The approximating functions must form a complete set in the domain of interest. Polynomials are used for ease of manipulation:

$$\begin{aligned} \theta(\eta, t) &= \sum_{i=1}^{m} b_i(t) \Phi_i^1(\eta) \\ v(\eta, t) &= \sum_{i=1}^{m} a_i(t) \Phi_i^2(\eta) \end{aligned} \tag{21}$$

where

$$\Phi_i^1(\eta) = \eta^{i-1}\left(\eta^2 - \frac{1}{4}\right)$$

$$\Phi_i^2(\eta) = \eta^{i-1}\left(\eta^2 - \frac{1}{4}\right)^2$$

The numerical procedure involved for the determination of a stability criterion has been covered thoroughly by previous authors. The operations involved are essentially as follows: formation of equation residuals, orthogonalization of the residuals with respect to the approximating functions Φ_i, and determination of the 2 m eigenvalues (complex) of a 2 m by 2 m matrix (complex). The coefficients of this matrix contain the dimensionless groups of the problem and the disturbance wave numbers. These are varied to determine the point where one of the eigenvalues has a positive real part. This determines the point of onset of instability to small disturbances. The existence of an eigenvalue with a positive real part is a sufficient condition for instability, but not a necessary one unless it is proved that the eigenfunctions form a complete set in the domain. Very little work for non self-adjoint systems has been done in this area.

NUMERICAL RESULTS AND DISCUSSION

The disturbances in the mean flow (x, y) and transverse (y, z) directions were treated separately. The combined results of the two disturbance directions then give a good estimate of the overall stability of the flow. The disturbances which yield the smaller critical Rayleigh number are those which give rise to the initial instabilities in the system. The primary criterion for convergence in the numerical treatment was the behavior of the critical and/or neutral Rayleigh and wave numbers. If a change of less than 1% was noted when additional approximating terms were added in the expansion, convergence was assumed. The smallest eigenvalues were also monitored for confirmation of this convergence. It should be noted here that the system behaved well for all transverse wave numbers but only for small and moderate mean flow wave numbers. For mean flow wave numbers approaching 15 to 20, the eigenvalues did not converge when twenty terms were used in the approximating functions.

The behavior of the problem was such that a different number of approximating terms was required for each disturbance direction. For the case of a Newtonian fluid with no dissipation, the analytical results of Chandrasekhar (1961) were used as a check on convergence for the transverse direction, and the numerical results of Deardorff (1965) were used for the mean flow direction. For the transverse disturbance case, three terms in the expansion produced convergence (in the above sense) with the analytical results. On the other hand, it was found that a six or seven term expansion was needed to give convergence to within 1% for the mean flow disturbances. In either direction, addition of nonzero Brinkman number or non-Newtonian parameters had only a small adverse effect on the convergence.

TABLE 1. NEWTONIAN FLUID RESULTS. CRITICAL RAYLEIGH NUMBER IS ONLY FOR THE MODE BEING CONSIDERED

Run	Re	Pr	Br	Ra_c	ν_{xc}	ν_{zc}
1	1	100	0	1707.911	—	3.124
2	500	10	0	1707.912	—	3.124
3	10	10	8	1528.310	—	3.292
4	10	10	0	5151.480	2.838	—
5	10	10	8	3100.319	3.085	—
6	200	0.1	0	3841.927	2.021	—
7	100	1	0	7672.035	1.823	—

TABLE 2. EFFECT OF BRINKMAN NUMBER ON EXCHANGE OF STABILITIES (OVERSTABILITY)

$Re = 10$
$Pr = 10$
$\lambda = 0.001$
$\nu_x = 4.5$

Br	$Im(\sigma)$	Ra	A
0	-0.00001	10,000	0.0
10	-0.0083	9,000	0.0
5	-0.0046	10,000	0.0
10	-0.01167	10,000	0.5
5	-0.00584	10,000	0.5

$Re = 1$
$Pr = 100$
$\lambda = 0.001$

Br	$Im(\sigma)$	Ra	A	ν_x
0	-0.00005	2500	1	4.0
1	-0.1571	2600	1	4.0
10	-0.6965	2500	1	4.0

Newtonian Fluids

For the Newtonian fluid, under the conditions tested, the transverse disturbances are most destabilizing. Different values of the physical parameters may yield mean flow disturbance dominance, however. The Brinkman number has a destabilizing effect on the flow in all cases treated. For zero Brinkman number, the exchange of stabilities hypotheses held within the range of parameters studied. As the dissipation terms were included (nonzero Brinkman number), overstability resulted in all instances. The growth frequency magnitude increased with the Brinkman number. Some of the Newtonian fluid results can be seen in Tables 1 and 2.

Non-Newtonian Fluids

The non-Newtonian fluid case was examined for values of the Reynolds number and Prandtl number of $Re = 10$, $Pr = 10$ (Case 1) and $Re = 1$, $Pr = 100$ (Case 2). For polymer melts the low Reynolds number, high Prandtl number regime is of most interest. The Brinkman number was varied between 0 and 10, and the first normal stress parameter A was examined between 0 and 1.5. With the other non-Newtonian stress parameters set to zero, A corresponds to the quantity $N_1/\overline{\tau_{xy}}$ (N_1 is the first normal stress difference). This is normally referred to as the Weissenberg number and is a measure of the relative importance of elastic to viscous stress effects.

Figure 3 shows the effects of Brinkman number and first normal stress difference on the neutral stability curves for the mean flow case. Both parameters have the effect of flattening out the curve as well as destabilizing the flow. The critical wave number increases as the flow becomes less stable. The critical Rayleigh number has been defined as the minimum neutral Rayleigh number obtained by

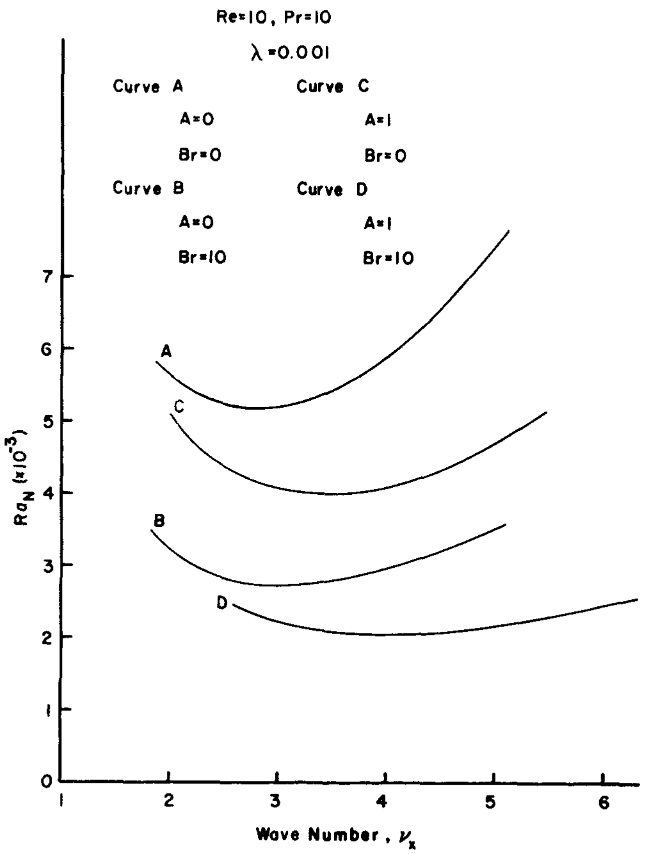

Fig. 3. Neutral Rayleigh Number as a function of mean Flow Wave Number for various values of Brinkman Number and Weissenberg Number (A). Non-Newtonian parameters *B, C* and *D* have been set equal to zero.

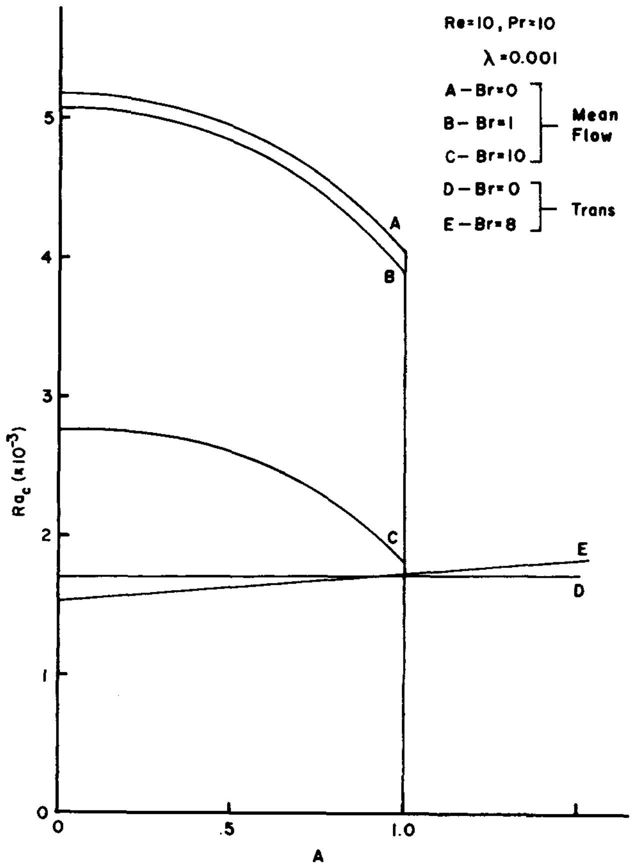

Fig. 4. Critical Rayleigh Number as a function of Weissenberg Number (A) for Mean Flow and Transverse Disturbances (Case 1). Non-Newtonian parameters *B, C* and *D* have been set equal to zero.

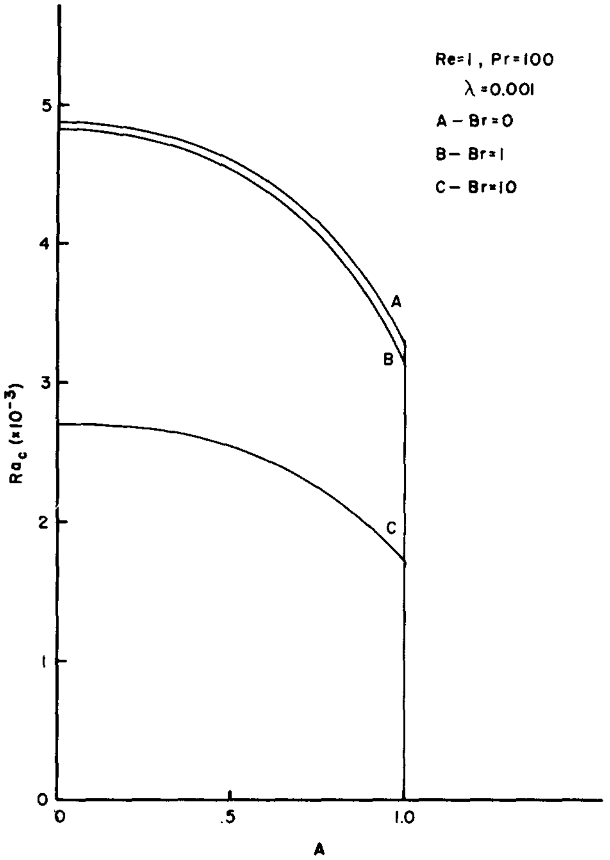

Fig. 5. Critical Rayleigh Number as a Function of Weissenberg Number (A) for Mean Flow Disturbances (Case II). Non-Newtonian parameters *B, C* and *D* have been set equal to zero.

varying the disturbance wavelength, holding all other physical parameters constant.

As in the Newtonian fluid case, the situation of zero Brinkman number shows exchange of stabilities for both disturbance directions. Green (1968) and Vest and Arpaci (1969) have indicated that overstability is possible for viscoelastic fluids in the Benard problem, if the fluid possesses an unusual combination of material properties: high elasticity and low viscosity. The addition of the dissipation term yields overstability, with the growth frequency magnitude increasing with the Brinkman number (Table 2). The effect of Brinkman number on the critical Rayleigh number can be seen in Figures 4 and 5. The Brinkman number has a destabilizing effect on the mean flow component for both cases shown. The effect of dissipation terms on the transverse stability is somewhat more complicated. For small values of A, the Brinkman number destabilizes the flow. However, as A approaches 1.0, the effect is reversed, and the dissipation terms slightly stabilize the flow.

Figures 4 and 5 also show the effect of the normal stress parameter A on the critical Rayleigh number at fixed Brinkman number. For values of A from 0 to 1, the mean flow is modestly destabilized by increasing this parameter. When A is greater than 1, an instability is observed in the mean flow case with zero critical Rayleigh number. This is a purely rheological instability, as a zero Rayleigh number implies a flow with no temperature gradient, and isothermal plane Couette flow of a Newtonian fluid is known to be stable to infinitesimal disturbances for all Reynolds numbers. Figure 6 shows the neutral curves for $A = 1$ and 1.05 to amplify this point. This rheological instability occurs at an A value slightly below that normally observed experimentally for melt fracture. The present case deals with infinitesimal disturbances, however, and these would require a very large growth rate to cause melt fracture in a system of finite length. After emergence from a finite system, the growth rate would have to be large enough to overcome the relaxation following the cessation of shearing. The value of A where the instability occurs is independent of Reynolds number, Brinkman number, and Prandtl number. No similar catastrophic rheological instability occurs for transverse disturbances.

As mentioned previously, the system is only numerically well behaved for small and moderate wave numbers ($<$ 20) in the flow direction. For higher wave numbers, a larger expansion was needed. The use of more than fifteen terms in the expansion resulted in the manipulation of nearly singular matrices. The problem of high wave numbers was then compounded by the numerical problems involved in handling nearly singular matrices. For this reason the behavior for high wave numbers was not ascertained. The rheological instability mentioned previously is evident at low enough wave numbers that it is believed to be genuine.

CONCLUSIONS

The inclusion of viscous dissipation effects in the problem of simple shearing flow with a superposed temperature gradient leads to overstability for both Newtonian and viscoelastic non-Newtonian fluids. This greatly complicates the numerical exploration of the effect of material parameters on the stability of the system. The disturbance frequency at neutral stability increases with increasing Brinkman number. For mean flow disturbances, increasing Brinkman number destabilizes the flow.

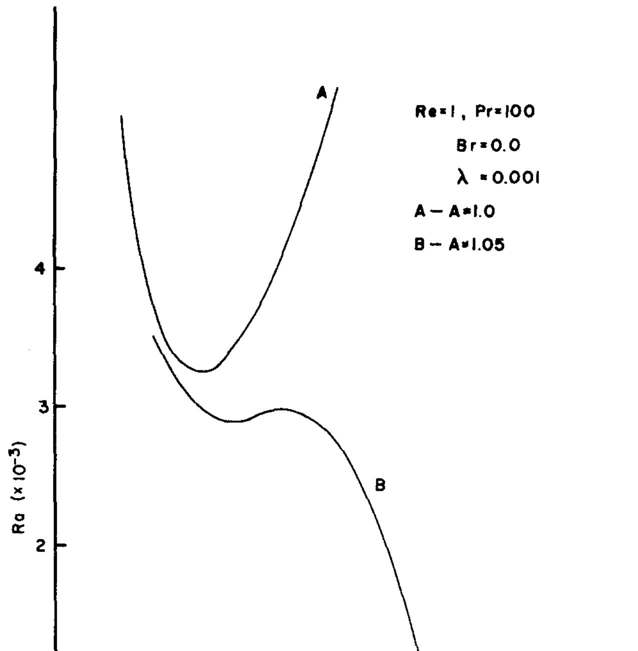

Fig. 6. Neutral Rayleigh Number as a Function of Mean Flow Wave Number—near "critical" value of Weissenberg Number (A). Non-Newtonian parameters *B, C* and *D* have been set equal to zero.

At a critical value of the Weissenberg number (N_1/τ_{xy}) of nearly 1, isothermal plane Couette flow exhibits an instability which is not present in Newtonian fluids. The critical value is nearly independent of Reynolds, Prandtl, and Brinkman numbers and is seen only if disturbances which are dependent on the flow direction are investigated. This gives further confirmation of the preliminary results of McIntire (1972) and Rothenberger et al. (1973) on a possible explanation of the phenomenon of polymer melt fracture, as the critical value of Weissenberg number found theoretically is very close to that observed in experimental melt flows.

The nature of the instability is vastly different from the convective instabilities generated for values of the Weissenberg number smaller than 1. This can be seen by examining Figure 6. The convective (Rayleigh) instability is characterized by a relatively long wavelength (small critical wave number) perturbations. The rheological (Weissenberg) instability is characterized by an extremely small wavelength disturbance (large critical wave number). This finding is in agreement with experimental results, which use the development of a short wavelength matte on fiber surfaces as the criterion for the onset of melt fracture instabilities.

Recently, Boger and Murthy (1972) and Boger and Halmos (1974) have shown experimentally that flows of polymer solutions through abrupt contractions develop a hydrodynamic instability when the ratio of the friction velocity to the shear wave velocity (Porteous and Denn, 1971) approaches 1. For simple shearing flows, this ratio can be shown to be essentially equal to the square root of the Weissenberg number (the difference being a numerical coefficient of order 1). Thus, the results given in this paper may actually be useful for quantitatively explaining the development of instabilities generated in either the die entry region or in the capillary die, leading to the melt fracture phenomenon.

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NOTATION

A = material function equivalent to the Weissenberg number
B, C, D = material functions defined in Equations (17) and (18)
Br = Brinkman number
C_v = specific heat
d = plate spacing
g = gravitational acceleration
K^2 = $\nu_x^2 + \nu_z^2$
m_p = material function defined in Equation (8)
N_1 = first normal stress difference
p = pressure
Pr = Prandtl number
$\mathbf{q}$ = heat flux vector
Ra = Rayleigh number
Re = Reynolds number
t = time
T = temperature
U = relative velocity of the plates
$\mathbf{u}$ = velocity disturbance vector
$\mathbf{v}$ = velocity vector
$v(\eta)$ = disturbance velocity amplitude function

Greek Letters

α = coefficient of thermal expansion
β_T = temperature gradient (without dissipation effects)
$\dot{\gamma}$ = shear rate
∇ = gradient operator
ξ, η, ζ = dimensionless spatial coordinates defined in Equation (14)
$\eta_p, \lambda_{1p}, \lambda_{2p}, \delta$ = material constants in Bird-Carreau model
σ = complex growth rate of perturbation function
ρ = density
θ = perturbation of the steady state temperature field
$\boldsymbol{\tau}$ = extra stress tensor
μ_i, β_o, β_1 = material functions defined in Equations (10) and (13)
ν_x, ν_z = disturbance wave numbers
Φ = Galerkin approximation functions
$II(t')$ = second invariant of the rate of strain tensor

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Phase Separation of Primary Dispersions in Beds Packed with Spherical Packings

Studies have been made of the behavior of drops in primary dispersions flowing through beds packed with glass ballotini in order to establish the mechanism of phase separation. The buoyancy and surface forces have been analyzed in terms of drop size and shape in the interstice in the packing and the physical properties of the dispersion. A mathematical model has been developed to describe drop behavior in the bed, and the drop size in the effluent dispersion has been predicted from established correlations.

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